UB-ECM-PF-04/11

Energy distribution in a BTZ black hole spacetime

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Abstract

We evaluate the energy distribution associated with the $(2+1)$-dimensional rotating BTZ black hole. The energy-momentum complexes of Landau-Lifshitz and Weinberg are employed for this computation. Both prescriptions give exactly the same form of energy distribution. Therefore, these results provide evidence in support of the claim that, for a given gravitational background, different energy-momentum complexes can give identical results in three dimensions, as it is the case in four dimensions.

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Introduction

One of the oldest problems in gravitation which still lacks of a definite answer is the localization of energy and momentum. Much attention has been devoted for this problematic issue. Einstein was the first to construct a locally conserved energy-momentum complex [1]. Consequently, a plethora of different energy-momentum complexes were proposed [2]-[7]. These expressions were restricted to evaluate energy distribution in quasi-Cartesian coordinates. Møller [8] proposed a new expression for an energy-momentum complex which could be utilized to any coordinate system. However, the idea of the energy-momentum complex was severely criticized for a number of reasons. Firstly, although a symmetric and locally conserved object, its nature is nontensorial and thus its physical interpretation seemed obscure [9]. Secondly, different energy-momentum complexes could yield different energy distributions for the same gravitational background [10, 11]. Thirdly, energy-momentum complexes were local objects while there was commonly believed that the proper energy-momentum of the gravitational field was only total, i.e. it cannot be localized [12]. For a long time, attempts to deal with this problematic issue were made only by proposers of quasi-local approach [13, 14].
In 1990 Virbhadra revived the interest in this approach [15]. At the same time Bondi [16] sustained that a nonlocalizable form of energy is not admissible in relativity so its location can in principle be found. Since then, numerous works on evaluating the energy distribution of several gravitational backgrounds have been completed employing the abandoned for a long time approach of energy-momentum complexes [17].
In 1996 Aguirregabiria, Chamorro and Virbhadra [18] showed that five different[2] energy-momentum complexes yield the same energy distribution for any Kerr-Schild class metric. Additionally, their results were identical with the results of Penrose [20] and Tod [21] using the notion of quasi-local mass.
Later attempts to deal with this problematic issue were made (as already mentioned) by proposers of quasi-local approach. The determination as well as the computation of the quasilocal energy and quasilocal angular momentum of a $(2+1)$-dimensional gravitational background were first presented by Brown, Creighton and Mann [22]. Many attempts since then have been performed to give new definitions of quasilocal energy in General Relativity [23]. Considerable efforts have also been performed in constructing superenergy tensors [24]. Motivated by the works of Bel [25] and independently of Robinson [26], many investigations have been carried out in this field [27].

[2]Later on Virbhadra [19] came to know that Tolman's and Einstein's complexes which had been used in [18] were exactly the same (see footnote 1 in [19]).

In 1999 Chang, Nester and Chen [28] proved that every energy-momentum complex is associated with a Hamiltonian boundary term. Thus, the energy-momentum complexes are quasi-local and acceptable.

In this work the approach of energy-momentum complexes is implemented. The gravitational background under investigation is the $(2+1)$-dimensional rotating BTZ black hole [29]. We evaluate the energy confined in a "one-sphere" (S^1) of radius r_0 associated with the aforesaid background. Specifically, we are implementing the prescriptions of Landau-Lifshitz and Weinberg. The specific $(2+1)$-dimensional black hole background is described by two parameters: the mass M and the angular momentum (spin) J. It is locally Anti-de Sitter space and thus it differs from Schwarzschild and Kerr solutions in that it is asymptotically Anti-de Sitter instead of flat. Additionally, it has no curvature singularity at the origin.

The rest of the paper is organized as follows. In the next section we consider the concept of energy-momentum complexes in the context of General Theory of Relativity. In Section 2 we briefly present the $(2+1)$-dimensional BTZ black hole in which the energy distribution by using two different prescriptions is to be calculated. In the subsequent two sections using Landau-Lifshitz and Weinberg energy-momentum complexes, respectively, we explicitly compute the energy distribution contained in a "one-sphere" of fixed radius r_0. The results extracted from these two different prescriptions associated with the same gravitational background are identical. Finally, in Section 5 a brief summary of results and concluding remarks are presented.

1 Energy-Momentum Complexes

The conservation laws of energy and momentum for an isolated, i.e., no external force acting on the system, physical system in the Special Theory of Relativity are expressed by a set of differential equations. Defining T^{μ}_{ν} as the symmetric energy-momentum tensor of matter and of all non-gravitational fields, the conservation laws are given by

$$T^{\mu}_{\nu,\,\mu} \equiv \frac{\partial T^{\mu}_{\nu}}{\partial x^{\mu}} = 0 \tag{1}$$

where

$$\rho = T^{t}_{t} \qquad j^{i} = T^{i}_{t} \qquad p_{i} = -T^{t}_{i} \tag{2}$$

are the energy density, the energy current density, the momentum density, respectively, and Greek indices run over the spacetime labels while Latin indices run over the spatial coordinate values.

Making the transition from Special to General Theory of Relativity, one adopts a simplicity principle which is called principle of minimal gravitational coupling. As a result of this, the conservation equation is now written as

$$T^{\mu}_{\nu;\,\mu} \equiv \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}\,T^{\mu}_{\nu}\right) - \Gamma^{\kappa}_{\nu\lambda}T^{\lambda}_{\kappa} = 0 \tag{3}$$

where g is the determinant of the metric tensor $g_{\mu\nu}(x)$. The conservation equation may also be written as

$$\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}\,T^{\mu}_{\nu}\right) = \xi_{\nu} \tag{4}$$

where

$$\xi_{\nu} = \sqrt{-g}\Gamma^{\kappa}_{\nu\lambda}T^{\lambda}_{\kappa} \tag{5}$$

is a non-tensorial object. For $\nu = t$ this means that the matter energy is not a conserved quantity for the physical system[3]. From a physical point of view, this lack of energy conservation can be understood as the possibility of transforming matter energy into gravitational energy and vice versa. However, this remains a problem and it is widely believed that in order to be solved one has to take into account the gravitational energy. By a well-known procedure, the non-tensorial object ξ_{ν} can be written as

$$\xi_{\nu} = -\frac{\partial}{\partial x^{\mu}}\left(\sqrt{-g}\,\vartheta^{\mu}_{\nu}\right) \tag{6}$$

where ϑ^{μ}_{ν} are certain functions of the metric tensor and its first order derivatives. Therefore, the energy-momentum tensor of matter T^{μ}_{ν} is replaced by the expression

$$\theta^{\mu}_{\nu} = \sqrt{-g}\,(T^{\mu}_{\nu} + \vartheta^{\mu}_{\nu}) \tag{7}$$

which is called energy-momentum complex since it is a combination of the tensor T^{μ}_{ν} and a pseudotensor ϑ^{μ}_{ν} which describes the energy and momentum of the gravitational field. The energy-momentum complex satisfies a conservation law in the ordinary sense, i.e.,

$$\theta^{\mu}_{\nu,\,\mu} = 0 \tag{8}$$

and it can be written as

$$\theta^{\mu}_{\nu} = \chi^{\mu\lambda}_{\nu\ \ ,\lambda} \tag{9}$$

where $\chi^{\mu\lambda}_{\nu}$ are called superpotentials and are functions of the metric tensor and its first order derivatives.

It is evident that the energy-momentum complex is not uniquely determined by the condition that its usual divergence is zero since it can always been added to the energy-momentum complex a quantity with an identically vanishing divergence.

[3]It is possible to restore the conservation law by introducing a local inertial system for which at a specific spacetime point $\xi_{\nu} = 0$ but this equality by no means holds in general.

2 BTZ black hole

In 1992 Bañados, Teitelboim, and Zanelli discovered a black hole solution (known as BTZ black hole) in $(2+1)$ dimensions [29]. Till that time it was believed that no black hole solution exists in three-dimensional spacetimes [30]. Bañados, Teitelboim, and Zanelli found a vacuum solution to Einstein gravity with a negative cosmological constant.
The starting point was the action in a three-dimensional theory of gravity

$$S = \int d^3x\sqrt{-g}\left(R + \frac{2}{l^2}\right) \tag{10}$$

where the radius of curvature l is related to the cosmological constant by $\Lambda = -l^{-2}$.
It is straightforward to check that the Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(R + \frac{2}{l^2}\right) = 0 \tag{11}$$

are solved by the metric[4]

$$ds^2 = N^2(r)dt^2 - \frac{dr^2}{N^2(r)} - r^2\left(N^\phi(r)dt + d\phi\right)^2, \tag{12}$$

where the squared lapse $N^2(r)$ and the angular shift $N^\phi(r)$ are given by

$$N^2(r) = \frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\ , \qquad N^\phi(r) = -\frac{J}{2r^2} \tag{13}$$

with $-\infty < t < +\infty$, $0 < r < +\infty$, and $0 \leq \phi \leq 2\pi$. Since the metric (12) satisfies the Einstein's field equations with a negative cosmological constant (see (11)), the metric is locally Anti-de Sitter

$$ds^2 = \left(1 + \frac{r^2}{l^2}\right)dt^2 - \frac{dr^2}{\left(1 + \frac{r^2}{l^2}\right)} - r^2d\phi^2 \tag{14}$$

and it can only differ from Anti-de Sitter space in its global properties. The two constants M and J are the conserved quantities: mass and angular momentum, respectively. The lapse function $N(r)$ vanishes for two values of the radial coordinate r given by

$$r_\pm^2 = \frac{l^2}{2}\left[M \pm \sqrt{M^2 - \left(\frac{J}{l}\right)^2}\right]\ . \tag{15}$$

[4] The form of the BTZ metric in quasi-Cartesian coordinates can be obtained by making the transformations

$$\begin{aligned} x &= r\cos(\phi) \\ y &= r\sin(\phi)\ . \end{aligned}$$

The largest root, r_+, is the black hole horizon. It is evident that in order for the horizon to exist one must have

$$M > 0 \ , |J| \leq Ml \ . \tag{16}$$

Therefore, negative black hole masses are excluded from the physical spectrum. There is, however, an important exceptional case. When one sets $M = -1$ and $J = 0$, the singularity, i.e., $r = 0$, disappears. There is neither a horizon nor a singularity to hide. The configuration is again that of Anti-de Sitter space. Thus, Anti-de Sitter emerges as a "bound state", separated from the continuous black hole spectrum by a mass gap of one unit.

For the specific case of spinless ($J = 0$) BTZ black hole, the line element (12) takes the simple form

$$ds^2 = \left(\frac{r^2}{l^2} - M\right) dt^2 - \frac{dr^2}{\left(\frac{r^2}{l^2} - M\right)} - r^2 d\phi^2 \ . \tag{17}$$

As it is stated in the Introduction, metric (12) of the rotating $(2+1)$-dimensional BTZ black hole is not asymptotically (that is as $r \to \infty$) flat

$$ds^2 = dt^2 - dr^2 - d\phi^2 \ . \tag{18}$$

Specifically, the BTZ black hole metric (12) for large r becomes

$$ds^2 = \left(\frac{r}{l}\right)^2 dt^2 - \left(\frac{r}{l}\right)^{-2} dr^2 - r^2 d\phi^2 \tag{19}$$

which coincides with the asymptotic form of Anti-de Sitter metric.

Additionally, the singularity at $r = 0$ is much weaker than that of the Schwarzschild spacetime. The curvature scalars for the case of Schwarzschild black hole are of the form

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \sim \frac{M^2}{r^6} \ . \tag{20}$$

It is easily seen that the singularity of Schwarzschild black hole at $r = 0$ is manifested by the power-law divergence of curvature scalars. For the case of BTZ black hole, the curvature scalars are

$$R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \sim \frac{1}{l^4} \tag{21}$$

and thus the BTZ solution has at most a δ-function singularity at $r = 0$, since everywhere else the spacetime is of constant curvature [31].

3 Landau-Lifshitz's Prescription

For a three-dimensional background the energy-momentum complex of Landau-Lifshitz [2] is defined by

$$L^{\mu\nu} = \frac{1}{2\kappa} S^{\mu\kappa\nu\lambda}{}_{,\kappa\lambda} \tag{22}$$

where κ is the gravitational coupling constant. The Landau-Lifshitz's superpotential $S^{\mu\kappa\nu\lambda}$ is

$$S^{\mu\kappa\nu\lambda} = -g\left(g^{\mu\nu}g^{\kappa\lambda} - g^{\mu\lambda}g^{\kappa\nu}\right) \ . \tag{23}$$

The energy-momentum complex of Landau-Lifshitz is symmetric in its indices

$$L^{\mu\nu} = L^{\nu\mu} \ . \tag{24}$$

Energy and momentum in Landau-Lifshitz's prescription for a three-dimensional background are given by

$$P^{\mu} = \int\int L^{t\mu} dx^1 dx^2 \ . \tag{25}$$

In particular, the energy of a physical system in a three-dimensional background takes the form

$$E = \int\int L^{tt} dx^1 dx^2 \ . \tag{26}$$

It is important to underscore that all calculations in the Landau-Lifshitz's prescription have to be restricted to the use of quasi-Cartesian coordinates.

Our primary task is to explicitly compute the Landau-Lifshitz's superpotentials (23). For the $(2+1)$-dimensional rotating BTZ black hole (12) there are only thirty six non-zero Landau-Lifshitz's superpotentials (see Appendix). Substituting Landau-Lifshitz's superpotentials into equation (22), one gets the energy density distribution

$$L^{tt} = \frac{1}{\kappa r^2}\left(\frac{r^2}{l^2} - \frac{J^2}{4r^2}\right)\left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{-2} \ . \tag{27}$$

Therefore, if we substitute the expression of the energy density (27) into equation (26), we get

$$\begin{aligned} E_{ext} &= 2\pi \int_{r_0}^{\infty} L^{tt} r dr \\ &= \frac{\pi}{\kappa}\left(\frac{r_0^2}{l^2} - M + \frac{J^2}{4r_0^2}\right)^{-1} \ . \end{aligned} \tag{28}$$

If we set the asymptotic value of the total gravitational mass of the system to be $M(\infty)$, then it is clear that the energy distribution associated with the rotating (2+1)-dimensional BTZ black hole under study, which is contained in a "one-sphere" of radius r_0, will be

$$E(r_0) = M(\infty) - \frac{\pi}{\kappa}\left(\frac{r_0^2}{l^2} - M + \frac{J^2}{4r_0^2}\right)^{-1} \quad . \tag{29}$$

A couple of comments are in order. Firstly, it is easily seen that for the case of the black hole horizon, i.e., $r_0 = r_+$, the energy distribution given by equation (29) becomes infinite. The infiniteness of the energy distribution stems from the infiniteness of the energy density given by equation (27) when we set r equal to r_+. However, this value for the radial coordinate should be excluded. It is known that it is meaningless the energy density of a closed system to be infinite and this is the case here since the energy density (27) is restricted to a "one-sphere" of radius r_0. Secondly, a neutral test particle experiences at a finite distance r_0 the gravitational field of the effective gravitational mass[5] described by expression (29). Thirdly, the energy-momentum complex of Landau-Lifshitz as formulated here for the rotating $(2+1)$-dimensional BTZ black hole satisfies the local conservation laws

$$\frac{\partial}{\partial x^\nu}\, L^{\mu\nu} = 0 \ . \tag{30}$$

4 Weinberg's Prescription

For a three-dimensional background the energy-momentum complex of Weinberg [7] is defined by

$$\tau^{\nu\lambda} = \frac{1}{2\kappa} Q^{\rho\nu\lambda}{}_{,\rho} \tag{31}$$

where Weinberg's superpotential $Q^{\rho\nu\lambda}$ is given by

$$Q^{\rho\nu\lambda} = \frac{\partial h^\mu_\mu}{\partial x_\nu}\eta^{\rho\lambda} - \frac{\partial h^\mu_\mu}{\partial x_\rho}\eta^{\nu\lambda} - \frac{\partial h^{\mu\nu}}{\partial x^\mu}\eta^{\rho\lambda} + \frac{\partial h^{\mu\rho}}{\partial x^\mu}\eta^{\nu\lambda} + \frac{\partial h^{\nu\lambda}}{\partial x_\rho} - \frac{\partial h^{\rho\lambda}}{\partial x_\nu} \ . \tag{32}$$

The symmetric tensor $h_{\mu\nu}$ reads

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \tag{33}$$

[5] Cohen and Gautreau were the first who gave a definition of effective mass by implementing Whittaker's theorem [32]. They derived the effective mass for a Reissner-Nordström black hole. Cohen and de Felice computed the total effective gravitational mass that a particle experiences while approaching the source of a Kerr-Newman spacetime [33]. Unfortunately, the expression for the total effective mass of the Kerr-Newman spacetime didn't incorporate the rotational contribution (setting the electric charge equal to zero gives the total mass M). R. Kulkarni, V. Chellathurai, and N. Dadhich generalized the result of Cohen and de Felice. They defined the total effective gravitational mass for a Kerr spacetime where the rotational effects were incorporated [34].

and $\eta^{\mu\nu}$ is the Minkowskian metric

$$(\eta^{\mu\nu}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \ . \tag{34}$$

It is important to stress that all indices on $h_{\mu\nu}$ and/or $\partial/\partial x_\mu$ are raised, or lowered, with the use of the Minkowskian metric. Additionally, the energy-momentum complex of Weinberg is symmetric in its indices

$$\tau^{\mu\nu} = \tau^{\nu\mu} \tag{35}$$

while the superpotential $Q^{\rho\nu\lambda}$ is antisymmetric in its first two indices

$$Q^{\rho\nu\lambda} = -Q^{\nu\rho\lambda} \ . \tag{36}$$

Energy and momentum in Weinberg's prescription for a three-dimensional background are given by

$$P^\nu = \int\int \tau^{\nu t} dx^1 dx^2 \ . \tag{37}$$

Specifically, the energy of a physical system in a three-dimensional background is

$$E = \int\int \tau^{tt} dx^1 dx^2 \ . \tag{38}$$

It should be underscored again that all calculations in the Weinberg's prescription have to be performed using quasi-Cartesian coordinates.
Since our aim is to evaluate the energy distribution associated with the rotating $(2+1)$-dimensional BTZ black hole background described by the line element (12), we firstly evaluate the Weinberg's superpotentials. There are sixteen non-zero superpotentials in the Weinberg's prescription (see Appendix). Substituting Weinberg's superpotentials into equation (31), the energy density distribution takes the form

$$\tau^{tt} = \frac{1}{\kappa r^2}\left(\frac{r^2}{l^2} - \frac{J^2}{4r^2}\right)\left(\frac{r^2}{l^2} - M + \frac{J^2}{4r^2}\right)^{-2} \tag{39}$$

It is evident that we have derived the same energy density distribution associated with the rotating $(2+1)$-dimensional BTZ black hole as in the case of the Landau-Lifshitz's prescription (see equation (27)). Thus, the energy distribution will be exactly the same with the one computed in Landau-Lifshitz's prescription (see equation (29)). The comments concerning the neutral test particle, the effective gravitational mass, and the local conservation laws made in the preceding section also hold here.

5 Conclusions

In this work we have explicitly evaluated the energy distribution, contained in a "one-sphere" of fixed radius r_0, of a rotating $(2+1)$-dimensional BTZ black hole. The gravitational background under consideration is an exact vacuum solution of Einstein's field equations in the presence of a negative cosmological constant. It is characterized by two conserved charges: the mass M and the angular momentum J. The energy distribution is obtained using two different energy-momentum complexes, specifically these are the energy-momentum complexes of Landau-Lifshitz and Weinberg. Both prescriptions give exactly the same energy distribution for the specific gravitational background. Consequently, the results obtained here support the claim[6] that for a given gravitational background, different energy-momentum complexes can give exactly the same energy and momentum distributions in three dimensions as they do in four dimensions. However, it should be stress that both Landau-Lifshitz's and Weinberg's energy-momentum complexes are symmetric (on the contrary Einstein's energy-momemtum complex is not symmetric) and do not depend on the second derivative of the metric.

It should also be pointed out that the energy distribution derived here can be regarded as the effective gravitational mass experienced by a neutral test particle placed in the rotating $(2+1)$-dimensional black hole background under consideration.

Setting the angular momentum J equals to zero, one can derive the energy distribution associated with the spinless $(2+1)$-dimensional BTZ black hole. Recently, I.-C. Yang and I. Radinschi [37] calculated the energy distributions associated with four $(2+1)$-dimensional black hole solutions by utilizing Einstein's and Møller's energy-momentum complexes. One of these spacetimes is the spinless $(2+1)$-dimensional BTZ black hole (or, the uncharged black hole solution as named in their manuscript). I.-C. Yang and I. Radinschi showed that the energy distribution of Einstein's energy-momentum complex is different from the one of Moller's energy-momentum complex. Furthermore, both expressions for the energy distribution associated with the spinless $(2+1)$-dimensional BTZ black hole are different from the one we have derived in the present analysis.

Finally, since the (2+1)-dimensional BTZ black hole is an asymptotically Anti-de-Sitter spacetime (AAdS), it would be an oversight not to mention that in the last years

[6]The claim that different pseudotensors give same results for local quantities was first stated by Virbhadra [15] and later on by Aguirregabiria, Chamorro, and Virbhadra in [18]. The author provided evidence in support of this claim for the case of a non-static spinless $(2+1)$-dimensional black hole with an outflux of null radiation [35]. However, this is not the case for the two-dimensional stringy black hole backgrounds. In particular, it was shown that Møller's energy-momentum complex provides meaningful physical results while the Einstein's energy-momentum complex fails to do so [36].

due to the AdS/CFT correspondence there has been much progress in obtaining finite stress energy tensors of AAdS spacetimes[7]. The gravitational stress energy tensor is in general infinite due to the infinite volume of the spacetime. In order to find a meaningful definition of gravitational energy one should subtract the divergences. The proposed prescriptions so far were ad hoc in the sense that one has to embed the boundary in some reference spacetime. The important drawback of this method is that it is not always possible to find the suitable reference spacetime. Skenderis and collaborators[8] [39, 43–45], and also Balasubramanian and Kraus [46], described and implemented a new method which provides an intrinsic definition of the gravitational stress energy tensor. The computations are universal in the sense that apply to all AAdS spacetimes. Therefore, it is nowadays right to state that the issue of the gravitational stress energy tensor for any AAdS spacetime has been thoroughly understood.

Acknowledgements

The author is indebted to K.S. Virbhadra for useful suggestions and comments on the manuscript. The author is also grateful to K. Skenderis for his enlightening comments on holographic renormalization method. The author has been supported by the European Research and Training Network "EUROGRID-Discrete Random Geometries: from Solid State Physics to Quantum Gravity" (HPRN-CT-1999-00161).

Appendix

Landau-Lifshitz's Superpotentials

Using equation (23) we can explicitly evaluate the superpotentials in the Landau-Lifshitz's prescription. There are thirty six non-vanishing superpotentials but for brevity we give the four superpotentials which are necessary to compute the energy density distribution (27)

$$S^{txtx} = \frac{J^2x^2 + 4\left(x^2+y^2\right)\left(-Mx^2+y^2+l^{-2}x^4+l^{-2}x^2y^2\right)}{\left(x^2+y^2\right)\left(J^2+4\left(x^2+y^2\right)\left(-M+l^{-2}\left(x^2+y^2\right)\right)\right)},$$

[7]For a short review see [38]

[8]Right after the first work of Henningson and Skenderis [39], Nojiri and Odintsov [40] calculated a finite gravitational stress energy tensor for an AAdS space where the dual conformal field theory is dilaton coupled. Furthermore, Nojiri and Odintson [41], and Ogushi [42] found well-defined gravitational stress energy tensors for AAdS spacetimes in the framework of higher derivative gravity and of gauged supergravity with single dilaton respectively.

$$
\begin{aligned}
S^{txty} &= \frac{xy\left(J^2+4\left(x^2+y^2\right)\left(-1-M+l^{-2}\left(x^2+y^2\right)\right)\right)}{\left(x^2+y^2\right)\left(J^2+4\left(x^2+y^2\right)\left(-M+l^{-2}\left(x^2+y^2\right)\right)\right)}\,, \\
S^{tytx} &= \frac{xy\left(J^2+4\left(x^2+y^2\right)\left(-1-M+l^{-2}\left(x^2+y^2\right)\right)\right)}{\left(x^2+y^2\right)\left(J^2+4\left(x^2+y^2\right)\left(-M+l^{-2}\left(x^2+y^2\right)\right)\right)}\,, \\
S^{tyty} &= \frac{4x^4\left(1+l^{-2}y^2\right)+4x^2y^2\left(1-M+2l^{-2}y^2\right)+y^2\left(J^2-4My^2+4l^{-2}y^4\right)}{\left(x^2+y^2\right)\left(J^2+4\left(x^2+y^2\right)\left(-M+l^{-2}\left(x^2+y^2\right)\right)\right)}\quad .
\end{aligned}
$$

Weinberg's Superpotentials

Using equation (32) we can explicitly evaluate the superpotentials in Weinberg's prescription. There are sixteen non-vanishing superpotentials but for brevity we give the four superpotentials which are necessary to compute the energy density distribution (39).

$$
\begin{aligned}
Q^{txt} &= -\frac{x\left(J^2+4\left(x^2+y^2\right)\left(-1-M+l^{-2}\left(x^2+y^2\right)\right)\right)}{\left(x^2+y^2\right)\left(J^2+4\left(x^2+y^2\right)\left(-M+l^{-2}\left(x^2+y^2\right)\right)\right)}\,, \\
Q^{tyt} &= -\frac{y\left(J^2+4\left(x^2+y^2\right)\left(-1-M+l^{-2}\left(x^2+y^2\right)\right)\right)}{\left(x^2+y^2\right)\left(J^2+4\left(x^2+y^2\right)\left(-M+l^{-2}\left(x^2+y^2\right)\right)\right)}\,, \\
Q^{xtt} &= \frac{x\left(J^2+4\left(x^2+y^2\right)\left(-1-M+l^{-2}\left(x^2+y^2\right)\right)\right)}{\left(x^2+y^2\right)\left(J^2+4\left(x^2+y^2\right)\left(-M+l^{-2}\left(x^2+y^2\right)\right)\right)}\,, \\
Q^{ytt} &= \frac{y\left(J^2+4\left(x^2+y^2\right)\left(-1-M+l^{-2}\left(x^2+y^2\right)\right)\right)}{\left(x^2+y^2\right)\left(J^2+4\left(x^2+y^2\right)\left(-M+l^{-2}\left(x^2+y^2\right)\right)\right)}\quad .
\end{aligned}
$$

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